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On complete subgraphs of color-critical graphs

Xiang-Ying Su

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

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Abstract

A graph G is called k-critical if $\chi(G) = k$ and $\chi(G - e) < \chi(G)$ for each edge e of G, where χ denotes the chromatic number. T. Gallai conjectured that every k-critical graph of order n contains at most n complete $(k-1)$-subgraphs. In 1987, Stiebitz proved Gallai's conjecture in the case $k = 4$, and in 1992 Abbott and Zhou proved Gallai's conjecture for all $k \geqslant 5$. In their paper, Abbott and Zhou asked the following question: is it true that the number of complete $(k-1)$-subgraphs of any k-critical graph G of order $n > k$ is at most $n - k + 3$ $(k \geqslant 5)$? In this paper, we give a positive answer to the question above for the cases $k = 5, 6$.

1. Introduction

We use standard notation. All graphs considered are finite, undirected and have neither loops nor multiple edges.

A graph G is called k-critical if $\chi(G) = k$ and $\chi(G - e) < \chi(G)$ for each edge e of G. For any graph G, we use $T_{k-1}(G)$ to denote the set of all complete $(k-1)$-subgraphs of G and $t_{k-1}(G)$ to denote the number of complete $(k-1)$-subgraphs of G; namely, $t_{k-1}(G) = |T_{k-1}(G)|$. Let C_l be the cycle of length l and K_d the complete d-graph. Then $W(l,d)$ denotes the graph obtained from C_l and K_d by joining each vertex of C_l to each vertex of K_d. We call $W(l,d)$ a d-wheel so that a 1-wheel is a wheel in the ordinary sense. We use $A \subset B$ to denote that A is a proper subset of B.

T. Gallai conjectured that $t_{k-1}(G) \leqslant n$ for every k-critical graph G of order n. The case $k = 3$ is trivial. In 1987, Stiebitz [3] proved Gallai's conjecture for the case $k = 4$. In 1992, Abbott and Zhou [1] proved the following theorem that is an extension of Gallai's conjecture.

Theorem 1 (Abbott and Zhou [1]). *Let G be a k-critical graph of order n. Then $t_{k-1}(G) \leqslant n$ with equality if and only if $k = n$ and $G = K_k$.*

In their paper, Abbott and Zhou asked the following question: is it true that the number of complete $(k-1)$-subgraphs of any k-critical graph G of order $n > k$ is at

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most $n - k + 3$ ($k \geqslant 5$)? In this paper, we prove some results concerning this question. Especially, we give an affirmative answer to this question for the cases $k = 5, 6$.

2. Main results

We need some linear algebra. Let G be a graph of order n. Consider the n-dimensional vector space $Z_2^{(n)}$ over the field Z_2. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then, for every subgraph H of G, there is exactly one corresponding vector $\alpha_H = (a_1, a_2, \ldots, a_n)$ where

$$a_j = \begin{cases} 1 & \text{if } v_j \in V(H), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if G is k-critical and has t complete $(k-1)$-subgraphs G_i, $i = 1, 2, \ldots, t$, then the corresponding vectors are simply denoted by α_i, $i = 1, 2, \ldots, t$, and the subspace of $Z_2^{(n)}$ spanned by α_i's is denoted by $S(G)$. The following lemma was proved by Abbott and Zhou [1].

Lemma 1. *If G is a k-critical graph of order n and G is not a $(k-3)$-wheel, then the dimension of the vector subspace $S(G)$ is equal to $t_{k-1}(G)$, the number of complete $(k-1)$-subgraphs of G.*

The proof of the following theorem is motivated by [1].

Theorem 2. *Let G be any k-critical graph of order $n > k$.*

(i) *If G has an edge that is not contained in any complete $(k-1)$-subgraph of G, then the number of complete $(k-1)$-subgraphs of G is at most $n - k + 2$, i.e., $t_{k-1}(G) \leqslant n - k + 2$.*

(ii) *If G has an edge that is contained in at most one complete $(k-1)$-subgraph of G, then the number of complete $(k-1)$-subgraphs of G is at most $n - k + 3$, i.e., $t_{k-1}(G) \leqslant n - k + 3$.*

Proof. Let G be any k-critical graph of order $n > k$. Assume that G contains t complete $(k-1)$-subgraphs G_i, $i = 1, 2, \ldots, t$. Let $S(G)$ be the subspace of $Z_2^{(n)}$ spanned by the vectors α_i's defined before. If $G = W(l, k-3)$ for some odd integer l, clearly G has an edge that is contained in exactly one complete $(k-1)$-subgraph of G and $t_{k-1}(G) = n - k + 3$. Hence we may assume that G is not a $(k-3)$-wheel. By Lemma 1, the subspace $S(G)$ is of dimension t. Let $S(G)^{\perp}$ denote the orthogonal complement of $S(G)$ in $Z_2^{(n)}$. Then $\dim(S(G)^{\perp}) = n - \dim(S(G)) = n - t$. Hence it is enough to show that $\dim(S(G)^{\perp}) \geqslant k - 2$ (resp. $\geqslant k - 3$) if G is a graph satisfying the condition of (i) (resp. of (ii)).

Proof of (i): Let e be an edge of G that is not contained in any complete $(k-1)$-subgraph of G. Since G is k-critical, $G-e$ is $(k-1)$-colorable. Let $V(G) = (V_1, V_2, \ldots, V_{k-1})$ be a coloring of $G-e$. Then, for any pair i,j, $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant k-1$, $|V(G_i) \cap V_j| = 1$ since e is not in G_i.

Now we define $k-2$ vectors $\beta_j = (b_{j1}, \ldots, b_{jn})$ as

$$b_{jr} = \begin{cases} 1 & \text{if } v_r \in V_j \text{ or } v_r \in V_{k-1}, \\ 0 & \text{otherwise,} \end{cases}$$

$j = 1, 2, \ldots, k-2$. Then the inner product is $\langle \alpha_i, \beta_j \rangle = 1 + 1 = 0$ for any pair i,j. Hence $\beta_j \in S(G)^\perp, j = 1, 2, \ldots, k-2$. To show that $\dim(S(G)^\perp) \geqslant k-2$, it is enough to show that $\beta_1, \ldots, \beta_{k-2}$ are linearly independent over Z_2. Suppose it is not so. Let $\beta_{j1}, \ldots, \beta_{jl}$ be a minimal dependent set. Then $\beta := \beta_{j_1} + \cdots + \beta_{j_l} = 0$. Clearly $l \geqslant 1$. Let v_r be a vertex in V_{j_1}. Then $b_{j_1 r} = 1$ and $b_{j_i r} = 0$ for all $2 \leqslant i \leqslant l$. It follows that the rth coordinate of β is 1, contrary to $\beta = 0$. Therefore $\beta_1, \ldots, \beta_{k-2}$ are independent and so $\dim(S(G)^\perp) \geqslant k-2$.

Proof of (ii): By (i), we may assume that G has an edge $e = uv$ that is contained in exactly one complete $(k-1)$-subgraph of G, say G_t. Since G is k-critical, $G-e$ has a proper $(k-1)$-coloring $V(G) = (V_1, V_2, \ldots, V_{k-1})$ such that u and v are in the same V_i, say V_{k-1}. By the assumptions, $|V(G_i) \cap V_j| = 1$ for each pair $i,j, 1 \leqslant i \leqslant t-1$, $1 \leqslant j \leqslant k-1$, and $|V(G_t) \cap V_{k-1}| = 2$, $|V(G_t) \cap V_j| = 0$ for some j and $|V(G_t) \cap V_{j'}| = 1$, for all $j' \neq j$, $1 \leqslant j' \leqslant k-2$. Without loss of generality we may assume that $j = k-2$. Define $k-3$ vectors $\beta_j = (b_{j1}, \ldots, b_{jn}), j = 1, 2, \ldots, k-3$, as

$$b_{jr} = \begin{cases} 1 & \text{if } v_r \in V_j \text{ or } v_r \in V_{k-3}, \\ 0 & \text{otherwise,} \end{cases}$$

$j = 1, 2, \ldots, k-4$, and

$$b_{k-3,r} = \begin{cases} 1 & \text{if } v_r \in V_{k-2} \text{ or } v_r \in V_{k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

By a similar way as in the proof of (i), one can verify that $\beta_1, \ldots, \beta_{k-3}$ form a linearly independent set of $S(G)^\perp$. Therefore $\dim(S(G)^\perp) \geqslant k-3$. □

Theorem 3. *If* $4 \leqslant k \leqslant 6$, *then any k-critical graph G of order greater than k has an edge that is contained in at most one complete $(k-1)$-subgraph of G.*

The proof of this theorem will be given in Section 3. Combining Theorems 2 and 3, we obtain at once the following result which gives a positive answer to Abbott and Zhou's question for the cases $k = 5, 6$.

Theorem 4. *If* $4 \leqslant k \leqslant 6$, *then any k-critical graph G of order $n > k$ contains at most $n-k+3$ complete $(k-1)$-subgraphs.* □

The bound $n - k + 3$ in Theorem 4 is attained by $W(l, k - 3)$, for any odd $l \geqslant 5$. We conjecture that in fact Theorem 3 is true for all k-critical graphs, $k \geqslant 4$.

Conjecture. Any k-critical graph G of order greater than k has an edge which is contained in at most one complete $(k - 1)$-subgraph of G, $k \geqslant 7$.

If one could prove this conjecture, Abbott and Zhou's question mentioned before would have a positive answer by Theorem 2. Moreover, it is of interest to know whether or not every k-critical graph G must have an edge which is not contained in any complete $(k - 1)$-subgraph if G is not a $(k - 3)$-wheel. If it is so, then by Theorem 2 we have $t_{k-1}(G) \leqslant n - k + 2$ whenever the k-critical graph G is not a $(k - 3)$-wheel and this bound would be the best possible as pointed out in [1].

3. Proof of Theorem 3

This section is dedicated to the proof of Theorem 3.

Let G any k-critical graph of order $n > k$, $4 \leqslant k \leqslant 6$. The case $k = 4$ was proved in [1, p.227].

Now consider the case $k = 5$. Suppose that G is a counterexample. For each edge e of G, let $t(e; G)$ denote the number of complete 4-subgraphs of G containing e. Then $t(e; G) \geqslant 2$ for all $e \in E(G)$. Note that a complete 4-graph contains 6 edges. We have

$$6t_4(G) = \sum_{e \in E(G)} t(e; G) \geqslant 2|E(G)| \geqslant \delta(G)n,$$

where $\delta(G)$ is the minimum degree of G. By Theorem 1, $t_4(G) < n$ and so $\delta(G) < 6$. Note that G is 5-critical; hence $\delta(G) \geqslant 4$. Let u be a vertex of G with $d(u) = \delta(G)$. Then $d(u) = 4$ or $d(u) = 5$. Denote by $N(u)$ the neighbor set of u in G. We claim that the induced subgraph $G[N(u)]$ has the property: each vertex of $G[N(u)]$ is contained in at least two triangles (of $G[N(u)]$). In fact, for any $v \in N(u)$, if H_1 and H_2 are two distinct complete 4-subgraphs of G containing the edge uv, then $H_1 - u$ and $H_2 - u$ are two distinct triangles of $G[N(u)]$ containing the vertex v. Hence the number of triangles of $G[N(u)]$ containing v equals $t(uv; G)$ and so is greater than or equal to two. Furthermore, we claim that G has no vertex with degree $n - 1$. Suppose $v \in V(G)$ has degree $n - 1$. Then $G - v$ is a 4-critical graph of order $n - 1 > 4$ as $n > 5$. Hence by the case $k = 4$, $G - v$ has an edge e which is contained in at most one triangle of $G - v$ so that e is contained in at most one complete 4-subgraph of G, a contradiction. Therefore $d(v) < n - 1$ for each $v \in V(G)$. So, $G[N(u) \cup \{u\}]$ is a proper subgraph of G. Since G is 5-critical, $G[N(u) \cup \{u\}]$ is 4-colorable and so $G[N(u)]$ is 3-colorable, i.e., $\chi(G[N(u)]) = 3$ (note that $G[N(u)]$ has triangles). Our purpose is to show that $G[N(u)]$ contains an edge $e = xy$ such that any proper 3-coloring of $G[N(u)] - e$ must assign the two end vertices x and y with different colors. Then, since G is 5-critical, $G - e$ is 4-colorable and, moreover, if c is a proper 4-coloring of $G - e$, then

c induces a proper 3-coloring of $G[N(u)] - e$ such that x and y have the same color, a contradiction.

Now if $d(u)$ is 4, one can verify easily that $G[N(u)]$ is a complete 4-graph, a contradiction. So we assume that $d(u) = 5$ and that $N(u) = \{v_1, \ldots, v_5\}$. Let H be a triangle of $G[N(u)]$, say $V(H) = \{v_1, v_2, v_3\}$. Then $v_4 v_5$ must be an edge of $G[N(u)]$. For otherwise, since v_4 is contained in at least two triangles of $G[N(u)]$, v_4 is adjacent to all the three vertices v_i, $i = 1, 2, 3$, so that $G[N(u)]$ contains the complete 4-subgraph formed by v_j, $j = 1, \ldots, 4$, contrary to $\chi(G[N(u)]) = 3$. Hence $v_4 v_5 \in G[N(u)]$. Since v_4 is in two triangles of $G[N(u)]$, one of them must contain two vertices of H, say v_1 and v_2. Hence v_4 is adjacent to v_1 and v_2. Similarly, v_5 is adjacent to, say, v_2 and v_3. Let c be any proper 3-coloring of $G[N(u)] - v_4 v_5$. Without loss of generality, we may assume that $c(v_i) = i$, $i = 1, 2, 3$. Then v_4 and v_5 must be assigned with color 3 and color 1, respectively, a contradiction.

Next we deal with the case $k = 6$. Let G be a counterexample. Let $t(e; G)$ denote the number of complete 5-subgraphs of G containing the edge e. Then, since a complete 5-graph contains ten edges, we have

$$10 t_5(G) = \sum_{e \in E(G)} t(e; G) \geqslant 2|E(G)| \geqslant \delta(G) n > \delta(G) t_5(G),$$

by Theorem 1. It follows that $5 \leqslant \delta(G) < 10$. Let u be a vertex of G with $d(u) = \delta(G)$. Then $5 \leqslant |N(u)| \leqslant 9$. As in the case $k = 5$, for each vertex v of the induced subgraph $G[N(u)]$, v has the following property: v is contained in at least two complete 4-subgraphs (of $G[N(u)]$) and $\chi(G[N(u)]) = 4$. So, it is enough to prove the following lemma.

Lemma 2. *Let H be a 4-chromatic graph with $n \leqslant 9$ vertices and assume that every vertex of H is contained in at least two complete 4-subgraphs. Then H contains an edge $e = xy$ such that any proper 4-coloring of $H - e$ assigns different colors to x and y (we call such an edge e a* required *edge).*

Proof of Lemma 2. Let us start with some simple observations.

(S1) $d(x; H) \geqslant 4$ for every $x \in V(H)$.

(S2) If z is a vertex of degree 4 in H, then the subgraph of H induced by the set $N(z; H) \cup \{z\}$ is a K_5^-.

(S3) Let z be a vertex of degree 5 in H. If H does not contain a K_5^-, then $N(z; H)$ induces a subgraph of H whose edges form two triangles with exactly one vertex in common.

(S4) Let H' be a 4-chromatic subgraph of H, and let x, x' be two distinct vertices of H' such that $c'(x) = c'(x')$ for every proper 4-coloring c' of H'. If x and x' have a common neighbor $y \in V(H) - V(H')$, then both xy and $x'y$ are required edges of H.

(S5) If H contains an induced 4-cycle (x, y, x', z) such that z is a vertex of degree 4 in H, then both xy and $x'y$ are required edges of H (use (S2) and (S4)).

Now suppose on the contrary that H does not contain a required edge. To derive a contradiction, let us consider a longest sequence of pairwise distinct vertices, say $x_1, \dots, x_r$, such that, for $i = 1, \dots, r$, the subgraph H_i of H induced by $\{x_1, \dots, x_i\}$ is uniquely 4-colorable (this means, in particular, that $H_4 = K_4$). Put $H^1 = H_r$, $H^2 = H - V(H^1)$, and $s = |V(H^2)|$. Let c^1 denote the (unique) 4-coloring of H^1. Then, we have

(S6) $r \geqslant 4$ and $d(x_r; H^1) = 3$ (use (S4) with $H' = H_{r-1}$). Hence $s \geqslant 1$ (see (S1)).

(S7) Let $y \in V(H^2)$. Then the neighbors of y in H^1 have pairwise distinct colors with respect to c^1 (use (S4)). Consequently, y has at most two neighbors in H^1 (otherwise, $x_1, \dots, x_r, y$ would induce an uniquely 4-colorable subgraph of H) and so $d(y; H^2) \geqslant 2$ (see (S1)).

By (S7) and (S6), we have $s \geqslant 3$, $s + r \leqslant 9$, and $4 \leqslant r \leqslant 6$. Now we distinguish three cases.

Case 1: $r = 6$. Then $s = 3$ and it is easy to see that $H^2 = K_3$, where each vertex of H^2 has degree 4 in H. From (S2) we then conclude that there are two nonadjacent vertices in H^1, say x and x', such that both x and x' are adjacent to all three vertices of H^2. Since H is 4-colorable, it follows that $c^1(x) = c^1(x')$, contradicting (S7).

Case 2: $r = 5$. Then $H^1 = K_5^-$, where x_1, x_2, x_3 form a triangle and x_4, x_5 are nonadjacent but completely joined to x_1, x_2, x_3. In particular, $d(x_4; H^1) = d(x_5; H^1) = 3$ and $c^1(x_4) = c^1(x_5)$.

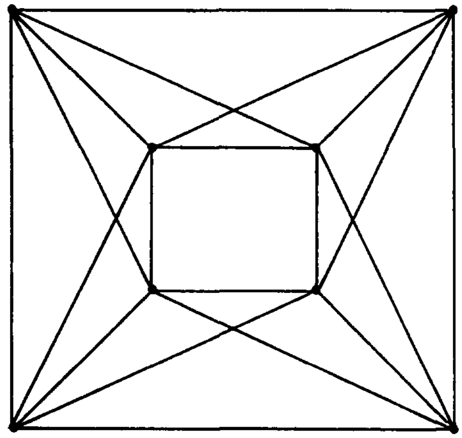
Suppose that x_4 (resp. x_5) has only one neighbor in H^2, say y. Then x_4 (resp. x_5) is a vertex of degree 4 in H, which implies (use (S2)) that y has three neighbors in H^1, contrary to (S7). Hence both x_4 and x_5 have at least two neighbors in H^2. Since x_4 and x_5 do not have a common neighbor in H^2 (see (S7)) and since $s \leqslant 4$, we conclude that $s = 4$ and H^2 consists of four vertices, say y_1, y_2, y_3, y_4, such that x_4 is adjacent to y_1 and y_2 but not to y_3 or y_4 and x_5 is adjacent to y_3 and y_4 but not to y_1 or y_2.

Since the graph H^2 has minimum degree $\geqslant 2$, we have $H^2 \in \{C_4, K_4^-, K_4\}$. If H^2 is an induced 4-cycle, then each vertex of H^2 has degree 4 in H and so H contains a required edge (see (S5)), a contradiction. If $H^2 = K_4^-$, then $d(y; H^2) = 2$ for some $y \in V(H^2)$ and hence $d(y; H) = 4$, which implies (use (S2)) that y is adjacent to both x_4 and x_5, contradicting (S7). Therefore, H^2 is a complete graph on 4 vertices.

Note that both x_4 and x_5 are contained in at least two complete 4-subgraphs. Hence by (S7), there are two vertices $x, x' \in \{x_1, x_2, x_3\}$ such that x is adjacent to y_1 and y_2 and x' is adjacent to y_3 and y_4. Since H is 4-colorable, we have $x \neq x'$. Now it is easy to check that xy_1 is a required edge of H, a contradiction.

Case 3: $r = 4$. Then $H^1 = K_4$. By the choice of H^1, H does not contain a K_5^-. Hence $d(z; H) \geqslant 5$ for every $z \in V(H)$ (see (S1) and (S2)). Because of (S7), this implies that $d(y; H^2) \geqslant 3$ for every $y \in V(H^2)$ and so $4 \leqslant s \leqslant 5$.

If $s = 5$, then it is easy to see that H^2 contains no complete 4-subgraph. Note that H^2 cannot be a 3-regular graph. Hence H^2 must be a wheel $W(4, 1)$. Let z be some vertex of degree 3 in H^2. Then z has exactly two neighbors in H^1 and so $d(z; H) = 5$. Since H does not contain a K_5^-, it is easy to check (use (S3)) that z is contained in at most one complete 4-subgraph, a contradiction.



e

Fig. 1.

If $s = 4$, then $H^2 = K_4$. Since the graph H has minimum degree $\geqslant 5$ and does not contain a K_5^-, we obtain that each vertex of H^1 has exactly two neighbors in H^2 and each vertex of H^2 has exactly two neighbors in H^1. Now it is easy to check (use (S3)) that H is the graph given in Fig. 1. But then e is a required edge of H, a contradiction. $\square$

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